PHYSICS 232
2014/15 Term 2

Final Examination

Name: Jake Bobowski
Student Number: Sol'ns

Free Response: Write out complete answers to the following questions. Show your work.

(10pts) **1.** (a) Bits are sent over a communications channel in packets of 12. If the probability of a bit being corrupted over this channel is 0.1 and such errors are independent, what is the probability that no more than 2 bits in a packet are corrupted? (**5 marks**)

(b) If 6 packets are sent over the channel, what is the probability that at least one packet will contain 3 or more corrupted bits? (**5 marks**)

(a) Binomial dist'n $P_B(x;p,n) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$

$P_{x \leq 2} = P_0 + P_1 + P_2$

$n = 12$
$p = 0.1$
$x = 0, 1, 2$

$P_0 = \frac{12!}{0!\,12!}(0.1)^0(0.9)^{12} = (0.9)^{12} = 0.282$

$P_1 = \frac{12!}{1!\,11!}(0.1)^1(0.9)^{11} = 12(0.1)(0.9)^{11} = 0.377$

$P_2 = \frac{12!}{2!\,10!}(0.1)^2(0.9)^{10} = \frac{12 \cdot 11}{2}(0.1)^2(0.9)^{10} = 0.230$

$\boxed{\therefore\ P_{x \leq 2} = 0.889}$

10 pts

(b) $P_{x \geq 3} = 1 - P_{x \leq 2} = 1 - 0.889 = 0.111 \equiv p^*$

p^* is prob. that a packet has 3 or more corrupted bits.

for $N = 6$ packets

$$P_B = \frac{N!}{x!(N-x)!} (p^*)^N (1-p^*)^{N-x}$$

$$P_{x \geq 1} = 1 - P_0$$

$$P_0 = \frac{6!}{0!\,6!} (p^*)^0 (1-p^*)^6 = (1-0.111)^6 = 0.494$$

$$\boxed{\therefore P_{x \geq 1} = 1 - .494 = 0.506}$$

0 pts

(10[pts]) **2.** The Poisson distribution is given by:

$$P_P(x;\mu) = \frac{\mu^x}{x!}e^{-\mu}$$

(a) Determine $\langle x \rangle$, the mean value of x for this distribution. *Hint*: After writing down the appropriate sum and making some initial manipulations, make use of the substitution $y = x - 1$. **(5 marks)**

(b) Determine $\langle x^2 \rangle$, the mean value of x^2 for this distribution. *Hint*: After writing down the appropriate sum and making some initial manipulations, make use of the substitution $y = x - 1$ and the result of part (a). **(5 marks)**

For both parts (a) and (b), you must show all steps and all of your work. Writing down only the correct final answers will result in 1 out of a possible 5 marks for each part.

(a) $$\langle x \rangle = \sum_{x=0}^{\infty} x\, P_P(x;\mu) = \sum_{x=0}^{\infty} x \frac{\mu^x}{x!} e^{-\mu}$$

$$= 0 + \sum_{x=1}^{\infty} x \frac{\mu^x}{x!} e^{-\mu} = \sum_{x=1}^{\infty} \mu \frac{\mu^{x-1}}{(x-1)!} e^{-\mu}$$

let $y = x - 1$ when $x = 1,\ y = 0$

$x = \infty,\ y = \infty$

$$\therefore \langle x \rangle = \mu \sum_{y=0}^{\infty} \frac{\mu^y}{y!} e^{-\mu} = \mu \underbrace{\sum_{y=0} P_P(y;\mu)}_{1}$$

$$\therefore \boxed{\langle x \rangle = \mu}$$

10 pts

(b) $\langle x^2 \rangle = \sum_{x=0}^{\infty} x^2 P_P(x;\mu) = \sum_{x=0}^{\infty} x^2 \frac{\mu^x}{x!} e^{-\mu}$

$= 0 + \sum_{x=1}^{\infty} x^2 \frac{\mu^x}{x!} e^{-\mu} = \sum_{x=1}^{\infty} x \mu \frac{\mu^{x-1}}{(x-1)!} e^{-\mu}$

let $y = x-1$ when $x=1,\ y=0$
$x=\infty,\ y=\infty$ also $x = y+1$

$$\therefore \langle x^2 \rangle = \mu \sum_{y=0}^{\infty} (y+1) \underbrace{\frac{\mu^y}{y!} e^{-\mu}}_{P_P(y;\mu)}$$

$$= \mu \left[\underbrace{\sum_{y=0}^{\infty} y P_P(y;\mu)}_{\mu \text{ from (a)}} + \underbrace{\sum_{y=0}^{\infty} P_P(y;\mu)}_{1} \right]$$

$$\therefore \boxed{\langle x^2 \rangle = \mu(\mu+1)}$$

Extra:

By definition $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$
$= \mu(\mu+1) - \mu^2$
$= \mu^2 + \mu - \mu^2$
$= \mu$

$\therefore \sigma = \sqrt{\mu}$

0 pts

(10pts) **3.** The goal of this problem is to describe how to analyze a set of impedance versus frequency data to extract the resistance R and capacitance C of an RC-series circuit.

The impedance of a series RC circuit is given by:

$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

where $\omega = 2\pi f$ and f is frequency.

A researcher measures the current in the circuit that results from applying ac voltages at a number of different frequencies. From these data the frequency dependence of the impedance $|Z| = |v|/|i|$ is determined from 1 to 10 kHz as given in the table on the following page. Assume that the uncertainty in f is negligible.

f (kHz)	$\|Z\|$ (kΩ)	$\sigma_{\|Z\|}$ (kΩ)	x	y	σ_y
1.00	1800	100	1.00	3.24×10^6	3.6×10^5
2.00	870	50	0.25	7.57×10^5	8.7×10^4
5.00	580	40	0.040	3.36×10^5	4.6×10^4
7.50	510	40	0.0178	2.60×10^5	4.1×10^4
10.00	520	30	0.010	2.70×10^5	3.1×10^4

(a) Linearize the equation above such that the resistance R and capacitance C can be extracted from the slope and y-intercept of a straight line. Give the equation of the straight line and describe the plot (y vs x) that you would generate. What does y represent and what does x represent? How are R and C related to the y-intercept a and slope b of a linear fit to the data? (**6 marks**)

(b) Complete the three right-hand columns in the table above. That is, calculate the x, y, and σ_y values that you would plot. What are the units of x, y, and σ_y? (**4 marks**)

(a) $$|Z|^2 = R^2 + \left(\frac{1}{\omega C}\right)^2 = R^2 + \left(\frac{1}{2\pi f C}\right)^2$$

$$\therefore \underbrace{|Z|^2}_{y} = \underbrace{R^2}_{a} + \underbrace{\left(\frac{1}{2\pi C}\right)^2}_{b} \underbrace{\frac{1}{f^2}}_{x}$$

If plot $|Z|^2$ vs $\frac{1}{f^2}$ expect linear relationship.

$$y = a + b\,x$$

slope $b = \left(\frac{1}{2\pi C}\right)^2$ y-intercept $a = R^2$

10 pts

x is simply $\frac{1}{f^2}$

units of x are $\left(\frac{1}{kHz}\right)^2 = (ms)^2$

y is just $|Z|^2$

units of y are $(k\Omega)^2$

if $y = |Z|^2$

then $\sigma_y = \left| \frac{\partial y}{\partial |Z|} \sigma_{|Z|} \right| = 2|Z|\sigma_Z$

again, units of σ_y are $(k\Omega)^2$ .... same as y.

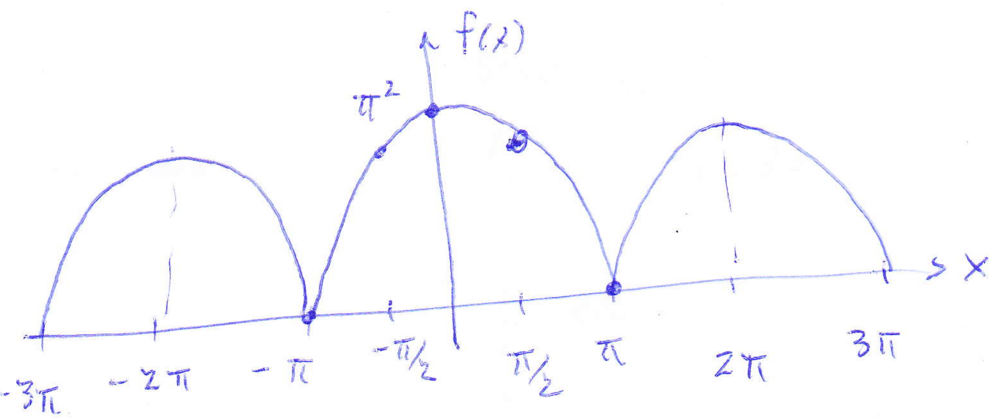
0 pts

(10pts) **4.** The function $f(x)$ is defined to be periodic with a period of 2π. On the interval $-\pi < x < \pi$ the function is defined as $f(x) = \pi^2 - x^2$.

(a) Sketch several periods of $f(x)$. Be sure to include scales for both the $x-$ and $y-$axes of your plot. (**2 marks**)

(b) Find the Fourier series for this function. Simplify your answers as much as possible. Write out the first five non-zero terms of the Fourier series. (**8 marks**)

x	$f(x) = \pi^2 - x^2$
$-\pi$	0
$-\pi/2$	$\pi^2 - \frac{\pi^2}{4} = \frac{3\pi^2}{4}$
0	π^2
$\pi/2$	$3\pi^2/4$
π	0



inverted parabola

(b) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\,dx = \frac{1}{\pi}\left[\pi^2 x - \frac{1}{3}x^3\right]\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[2\pi^3 - \frac{2\pi^3}{3}\right] = \frac{2}{3}2\pi^2 = \boxed{\frac{4\pi^2}{3} = a_0}$$

start w/ b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

<u>Note:</u> $f(x)$ is even (symmetric) fcn
$\sin nx$ is odd (antisymmetric) fcn
$\therefore$ prod. $f(x)\sin nx$ is odd fcn.

Integral of odd fcn over symmetric interval $-\pi \leq x \leq \pi$ is zero.

$$\therefore \boxed{b_n = 0 \quad \forall\, n}$$

10 pts

Now $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$f(x)$ & $\cos nx$ both even.
$\therefore f(x)\cos nx$ even too

$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

$= \frac{2}{\pi}\int_{0}^{\pi} (\pi^2 - x^2)\cos nx\,dx$

Start w/

$\int_0^{\pi} \pi^2 \cos nx\,dx$

$= \frac{\pi^2}{n}\sin nx \Big|_0^{\pi}$

$= \frac{\pi^2}{n}(\underbrace{\sin n\pi}_{0} - \underbrace{\sin 0}_{0}) = 0.$

$I_2 = -\int_0^{\pi} x^2 \cos nx\,dx$

Integrate by parts.

$u = x^2 \qquad dv = \cos nx\,dx$

$du = 2x\,dx \qquad v = \frac{1}{n}\sin nx$

$\int u\,dv = uv - \int v\,du$

$\therefore I_2 = -\int_0^{\pi} x^2\cos nx\,dx = \cancel{-\frac{x^2}{n}\sin nx\Big|_0^{\pi}}^{\,0} + \int_0^{\pi} \frac{2}{n}x\sin nx\,dx$

Integrate by parts again

$u = x \qquad dv = \sin nx\,dx$

$du = dx \qquad v = -\frac{1}{n}\cos nx$

0 pts

$I_2 = \frac{2}{n}\left[-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]$

Sol'ns

$$\therefore I_2 = \frac{2}{n}\left[-\frac{x}{n}\cos nx \Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx \, dx \right]$$

$$= \frac{2}{n}\left[+0 - \frac{\pi}{n}\cos(n\pi) + \cancel{\frac{1}{n^2}\sin nx \Big|_0^{\pi}}^{\,0} \right]$$

$$= \frac{-2\pi}{n^2} \underbrace{\cos(n\pi)}_{(-1)^n}$$

$$\therefore I_2 = \frac{-2\pi}{n^2}(-1)^n$$

$$\therefore a_n = \frac{2}{\pi}\left(\frac{-2\pi}{n^2}\right)(-1)^n = -\frac{4}{n^2}(-1)^n \qquad a_0 = \frac{4\pi^2}{3}$$

$$b_n = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

$$\boxed{\therefore f(x) = \frac{2\pi^2}{3} - 4\left(-\cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \frac{1}{16}\cos 4x - \dots \right)}$$

(10pts) **5.** If n measurements of a quantity x are made each with its own uncertainty σ (i.e. $x_1 \pm \sigma_1$, $x_2 \pm \sigma_2$, …, $x_n \pm \sigma_n$), then the appropriate weighted mean of the measurements is given by:

$$\mu = \frac{\sum_{i=1}^{n} \frac{x_i}{\sigma_i^2}}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}$$

(a) Find a general expression for the error in the weighted mean σ_μ. You must show all of your work. Simply writing down the correct final answer will earn only 1 out of a possible 5 marks. (**5 marks**)

(b) Three groups of particle physicists measure the mass of a certain elementary particle with the following results (in units of MeV/c^2): 1967.0 ± 1.0, 1969.0 ± 1.4, and 1972.1 ± 2.5. Find the weighted mean of these measurements and its uncertainty. (**5 marks**)

(a) $\mu = \mu(x_1, x_2, \ldots, x_n)$ use prop. of errors

$$\sigma_\mu^2 = \left(\frac{\partial \mu}{\partial x_1}\sigma_1\right)^2 + \left(\frac{\partial \mu}{\partial x_2}\sigma_2\right)^2 + \cdots + \left(\frac{\partial \mu}{\partial x_n}\sigma_n\right)^2$$

$$= \sum_{j=1}^{n}\left(\frac{\partial \mu}{\partial x_j}\sigma_j\right)^2$$

$$\frac{\partial \mu}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\frac{\sum_{i=1}^{n}\frac{x_i}{\sigma_i^2}}{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}\right] = \frac{1}{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n}\frac{x_i}{\sigma_i^2}\right)$$

$$= \frac{1}{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}\sum_{i=1}^{n}\left(\frac{1}{\sigma_i^2}\frac{\partial x_i}{\partial x_j}\right)$$

10 pts

note that $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$

$$\therefore \frac{\partial \mu}{\partial x_j} = \frac{1}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}} \underbrace{\sum_{i=1}^{n} \left(\frac{1}{\sigma_i^2} \delta_{ij} \right)}_{\frac{1}{\sigma_j^2}} = \frac{1/\sigma_j^2}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}$$

$$\therefore \sigma_\mu^2 = \sum_{j=1}^{n} \left(\frac{\partial \mu}{\partial x_j} \sigma_j \right)^2 = \sum_{j=1}^{n} \left(\frac{\frac{1}{\sigma_j}}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}} \right)^2$$

$$= \frac{1}{\left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{\cancel{2}}} \cancel{\left(\sum_{j=1}^{n} \frac{1}{\sigma_j^2} \right)}$$

$$\boxed{\therefore \sigma_\mu^2 = \frac{1}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}}$$

0 pts

Sol'ns

(6)

$$\mu = \frac{\sum_{i=1}^{3} \frac{x_i}{\sigma_i^2}}{\sum_{i=1}^{3} \frac{1}{\sigma_i^2}}$$

$x_1 = 1967.0 \pm 1.0$

$x_2 = 1969.0 \pm 1.4$

$x_3 = 1972.1 \pm 2.5$

$$\therefore \mu = 1968.10 \ \frac{MeV}{c^2}$$

$$\sigma_\mu^2 = \frac{1}{\sum_{i=1}^{3} \frac{1}{\sigma_i^2}} = 0.599$$

$$\therefore \sigma_\mu = 0.77 \ \frac{MeV}{c^2}$$

$$\boxed{\therefore \mu = 1968.1 \pm 0.8 \ \frac{MeV}{c^2}}$$

(10pts) **6.** In his famous experiment with electrons, J.J. Thompson measured the "charge-to-mass ratio" $r \equiv e/m$, where e is the electron's charge and m its mass. This experiment is done by accelerating electrons through a voltage V and then deflecting their direction in a magnetic field. The charge-to-mass ratio is given by:

$$r = \frac{125}{32\mu_0^2 N^2} \frac{D^2 V}{d^2 I^2}$$

The magnetic field is generated using a coil consisting of N loops where each loop has diameter D and carries current I. When it's deflected, the electron follows a curved path of radius d. If the experimentally measured quantities are:

$$D = 661 \pm 2 \text{ mm}$$
$$V = 45.0 \pm 0.2 \text{ V}$$
$$d = 91.4 \pm 0.5 \text{ mm}$$
$$I = 2.48 \pm 0.04 \text{ A}$$

what is the experimentally determined value of r and its uncertainty? Assume that $N = 72$ and $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 are both known exactly. (**8 marks**)

(b) The known values the electron charge and mass are $e = 1.60 \times 10^{-19}$ C and $m = 9.11 \times 10^{-31}$ kg. Does the experimentally determined value of r agree with the expected value? (**2 marks**)

(a) $$\sigma_r^2 = \left(\frac{\partial r}{\partial D}\sigma_D\right)^2 + \left(\frac{\partial r}{\partial V}\sigma_V\right)^2 + \left(\frac{\partial r}{\partial d}\sigma_d\right)^2 + \left(\frac{\partial r}{\partial I}\sigma_I\right)^2$$

$$= \left(2\frac{125}{32\mu_0^2N^2}\frac{DV}{d^2I^2}\sigma_D\right)^2 + \left(\frac{125}{32\mu_0^2N^2}\frac{D^2}{d^2I^2}\sigma_V\right)^2$$

$$+ \left(-2\frac{125}{32\mu_0^2N^2}\frac{D^2V}{d^3I^2}\sigma_d\right)^2 + \left(-2\frac{125}{32\mu_0^2N^2}\frac{D^2V}{d^2I^3}\sigma_I\right)^2$$

$$= \underbrace{\left(\frac{125}{32\mu_0^2N^2}\frac{D^2V}{d^2I^2}\right)}_{r}{}^2\left[\left(2\frac{\sigma_D}{D}\right)^2 + \left(\frac{\sigma_V}{V}\right)^2 + \left(2\frac{\sigma_d}{d}\right)^2 + \left(2\frac{\sigma_I}{I}\right)^2\right]$$

$$\boxed{\therefore\ \sigma_r = r\left[\left(2\frac{\sigma_D}{D}\right)^2 + \left(\frac{\sigma_V}{V}\right)^2 + \left(2\frac{\sigma_d}{d}\right)^2 + \left(2\frac{\sigma_I}{I}\right)^2\right]^{\frac{1}{2}}}$$

10 pts

$$r = \frac{125}{32\mu_0^2 N^2} \frac{D^2 V}{d^2 I^2} = 1.826 \times 10^{11} \frac{C}{kg}$$

$$\sigma_r = 6.4 \times 10^9 \frac{C}{kg}$$

$$\boxed{\therefore \; r = \frac{e}{m} = (1.83 \pm 0.06) \times 10^{11} \frac{C}{kg}}$$

(b) using $e = 1.60 \times 10^{-19} C$ $m = 9.11 \times 10^{-31} kg$

the expected value of r is $r = 1.756 \times 10^{11} \frac{C}{kg}$

meas. & expected values don't agree, but just barely!

0 pts

(10pts) **7.** In a certain dice game a player rolls a pair of dice. If the player rolls doubles, then he wins \$10, if he doesn't roll doubles and the pair of dice sum to an even number he must pay \$3, if he doesn't roll doubles and the pair of dice sum to an odd number he must pay \$1.50.

So, for example, if the player rolls a pair of 4's he wins \$10. If he rolls a 2 and a 4, he must pay \$3 because it's not a double and 2 + 4 is even. If he rolls a 2 and a 5, he must pay \$1.50 because it's not a double and 2 + 5 is odd.

(a) If the someone plays many many rounds of this game, what is the average amount of money won (or lost) per turn? (**4 marks**)

(b) What is the standard deviation in the amount paid out per turn? (**3 marks**)

(c) If someone plays the game exactly $N = 100$ times, what is the net amount of money (from all 100 rounds) that the player should expect to win (or lose)? What would be the uncertainty in that net amount? (**3 marks**)

	1	2	3	4	5	6
1	D +10	O −1.5	E −3	O −1.5	E −3	O −1.5
2	O −1.5	D +10	O −1.5	E −3	O −1.5	E −3
3	E −3	O −1.5	D +10	O −1.5	E −3	O −1.5
4	O −1.5	E −3	O −1.5	D +10	O −1.5	E −3
5	E −3	O −1.5	E −3	O −1.5	D +10	O −1.5
6	O −1.5	E −3	O −1.5	E −3	O −1.5	D +10

36 possible outcomes

6 doubles $P_D = \frac{6}{36} = \frac{1}{6}$

12 evens $P_E = \frac{12}{36} = \frac{1}{3}$

18 odds $P_O = \frac{18}{36} = \frac{1}{2}$

$$\langle D \rangle = \sum_{i=1}^{3} D_i P_i = (+10)\left(\frac{1}{6}\right) + (-3)\left(\frac{1}{3}\right) + (-1.5)\left(\frac{1}{2}\right)$$

$$\boxed{\therefore \langle D \rangle = -0.083}$$ lose money on avg

10 pts

$$\sigma^2 = \sum_{i=1}^{3} (D_i - \langle D \rangle)^2 P_i$$

$$= \sum_{i=1}^{3} \left(D_i^2 - 2\langle D \rangle D_i + \langle D \rangle^2\right) P_i$$

$$= \sum_{i=1}^{3} D_i^2 P_i - 2\langle D \rangle \underbrace{\sum_{i=1}^{3} P_i D_i}_{\langle D \rangle} + \langle D \rangle^2 \underbrace{\sum_{i=1}^{3} P_i}_{1}$$

$$\therefore \sigma^2 = \underbrace{\sum_{i=1}^{3} D_i^2 P_i}_{20.79} - \langle D \rangle^2$$

$$\boxed{\therefore \sigma = \$4.56}$$ large c.t. $\langle D \rangle$.

(c) if someone plays $N = 100$ times

expect to lose a net of $100(-0.083) = \boxed{\$-8.33}$

$\sigma_\mu = \dfrac{\sigma}{\sqrt{N}} = \dfrac{\sigma}{10} = 0.456$ Std. Error of mean of 100 trials

$D_{Net} = N \langle D \rangle$

large! could lose big or win big

$\therefore \sigma_{net} = N \sigma_\mu = N \dfrac{\sigma}{\sqrt{N}} = \sqrt{N} \sigma = \boxed{\$45.60}$

0 pts

(10pts) **8.** The goal of this problem is to estimate the value of a definite double integral of a function of two variables:

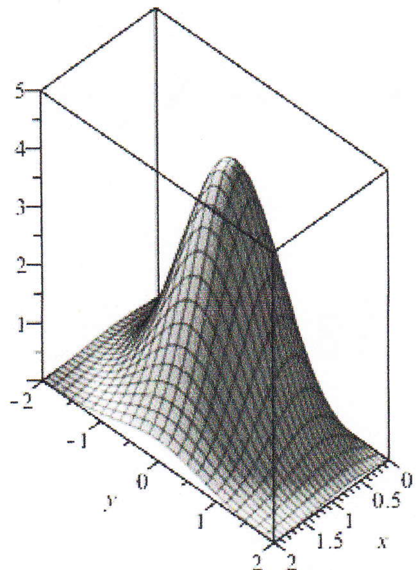
$$\int_{x_i}^{x_f} \int_{y_i}^{y_f} f(x,y)\, dy\, dx$$

using Monte Carlo methods.

For example, suppose that we want to evaluate the following integral of $f(x,y) = 10x\, e^{-x^2-y^2}$:

$$I = \int_0^2 \int_{-2}^{2} 10x\, e^{-x^2-y^2}\, dy\, dx$$

where the x- and y-integrals span $0 \le x \le 2$ and $-2 \le y \le 2$ respectively. A plot of $f(x,y)$ over these intervals is shown in the figure below.



(a) Describe in detail an implementation of the Monte Carlo f-average $(\bar{f})$ method that could be used to estimate the numerical value of the double integral of $f(x,y)$. **(6 marks)**

(b) Suppose that an f-average Monte Carlo method was implemented and it was found that $\bar{f} = 1.08 \pm 0.11$ over the plane spanned by $0 \le x \le 2$ and $-2 \le y \le 2$.

What would be the resulting estimate for the value of $I \pm \sigma_I$? **(4 marks)**

(a) 1. Generate uniformly dist^{d} x_i value between 0 & 2
2. Generate uniformly dist. y_i value between -2 & 2.
3. Evaluate fcn f_i at x_i & y_i point.
4. repeat many times $i = 1..N$ N large.
5. calculate $\bar{f} = \frac{1}{N} \sum_i f_i$

10 pts

The double integral finds the volume underneath the surface plot in the fig.

Want to find equiv. volume formed by box of height $\bar{f}$ & w/ base x width give by $0 \leq x \leq 2$ & $-2 \leq y \leq 2$ ranges.

i.e. want $$\iint f(x,y)\,dx\,dy = \bar{f}\underbrace{(2-0)}_{x\ \text{range}}\underbrace{(2-(-2))}_{y\ \text{range.}}$$

$$= 8\bar{f}.$$

or in general

$$\boxed{\int_{x_i}^{x_f}\int_{y_i}^{y_f} f(x,y)\,dy\,dx = \bar{f}\,(x_f - x_i)(y_f - y_i)}$$

(b) Know that $I = 8\bar{f}$

$\therefore\ \sigma_I = 8\,\sigma_{\bar{f}}$

If $\bar{f} = 1.08 \pm 0.11$, then $I = 8.64 \pm 0.88$

or $\boxed{I = 8.6 \pm 0.9}$

0 pts

(10pts) **9.** Suppose that we have a linear function $y = a + bx$ where a is the y-intercept and b is the slope. If we have a set of measurements $(x_i, y_i \pm \sigma_i)$ where $i = 1, 2, 3 \ldots, N$, then the best-fit values of a and b are determined from a minimization of χ^2:

$$\frac{\partial \chi^2}{\partial a} = 0$$
$$\frac{\partial \chi^2}{\partial b} = 0$$

(a) Write down an expression for χ^2 in terms of a, b, x_i, y_i, and σ_i. (**2 marks**)

(b) What is the expected value of χ^2 for a given set of data and best-fit parameters? Assume that $N \gg 2$. What does it mean if χ^2 turns out to be much larger than expected? What does it mean if χ^2 turns out to be much smaller than expected? (**3 marks**)

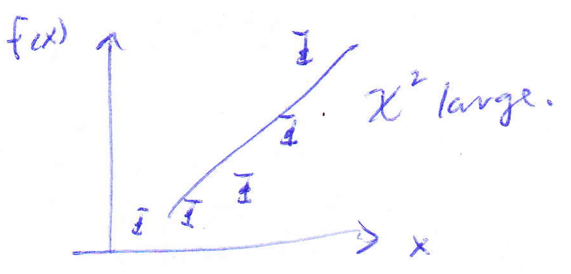
(c) Describe in detail the "method of maximum likelihood" used to determine the best-fit values of a and b from a set of measurements. You don't need to formally derive expressions for the best-fit values of a and b. Just start by writing down the probability of obtaining the data set $(x_1, y_1 \pm \sigma_1)$, $(x_2, y_2 \pm \sigma_2), \ldots,$ $(x_N, y_N \pm \sigma_N)$ for a given slope and y-intercept. Then argue that minimizing χ^2 as discussed above leads to the best values of a and b for the given dataset. (**5 marks**)

(a) $$\chi^2 = \sum_{i=1}^{N} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2 = \sum_{i=1}^{N} \left(\frac{y_i - a - bx_i}{\sigma_i} \right)^2$$

(b) on avg. expect deviation between y_i & $y(x_i)$ to be approx. equal to σ_i

$$\therefore \chi^2 = \sum_{i=1}^{N} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2 \simeq \sum_{i=1}^{N} \left(\frac{\sigma_i}{\sigma_i} \right)^2 = \sum_{i=1}^{N} 1 = N.$$

If find $\chi^2 \gg N$, then in means that data likely don't follow linear relationship

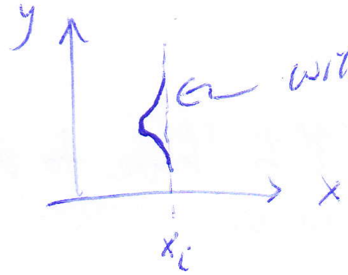


or possibly have underestimated σ_i values.

10 pts

If find $\chi^2 \ll N$ then have probably overestimated σ_i values $\rightsquigarrow$ dividing σ_i that's too big make χ^2 small.

(e) Imagine measuring y_i value over & over for fixed x_i



will get a dist'n of values w/ avg.
$y(x_i) = a + b x_i$ & std dev. σ_i
Dist'n will be Gaussian.

Prob. of meas y_i at x_i is given by $P_i = \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y_i - y(x_i)}{\sigma_i}\right)^2\right]$

or $$P_i = \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y_i - a - b x_i}{\sigma_i}\right)^2\right]$$

Prob. that will meas (x_1, y_1) followed by (x_2, y_2) followed by (x_3, y_3), ... is

$$P = P_1 P_2 P_3 \cdots P_N = \prod_{i=1}^{N} P_i = \prod_{i=1}^{N} \left\{ \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y_i - a - b x_i}{\sigma_i}\right)^2\right] \right\}$$

$$= \left[\prod_{i=1}^{N} \left(\frac{1}{\sigma_i\sqrt{2\pi}}\right)\right] \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{y_i - a - b x_i}{\sigma_i}\right)^2\right]$$

$$= \left[\prod_{i=1}^{N} \left(\frac{1}{\sigma_i\sqrt{2\pi}}\right)\right] \exp\left[-\frac{1}{2}\chi^2\right]$$

0 pts

Sol'ns

The method of maximum likelihood says that the best-fit values of a & b determined from the values that maximize P.

Equiv., need to find values that minimize χ^2 since have $e^{-\frac{1}{2}\chi^2}$.

$\therefore$ to find best-fit values of a & b have to minimize χ^2 w.r.t. a & b.

$$\frac{\partial \chi^2}{\partial a} = 0$$

$$\frac{\partial \chi^2}{\partial b} = 0$$